1 Exercises 4.1: 1(b), 2(c), 5(d)

1(b) Solve the heat equation \( u_t = 2u_{xx} \) for a rod of length \( L \) with both ends held at 0°, if \( L = 1, \ u(x, 0) = x \). (See Example 2, Section 3.6.)

Solution: We are solving the initial-boundary-value problem

\[
\begin{align*}
\frac{d}{dt}u & = 2\frac{d^2}{dx^2}u, \\
0 & < x < 1, \ t > 0, \\
& u(0, t) = u(1, t) = 0, \ t \geq 0, \\
& u(x, 0) = x, \ 0 \leq x \leq 1.
\end{align*}
\]

As before, we use separation of variables to solve, letting \( u(x, t) = X(x)T(t) \). Then, we obtain

\[
\begin{align*}
\frac{dT}{dt}X & = 2\frac{d^2X}{dx^2}T \\
\implies \frac{1}{2T}\frac{dT}{dt} & = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda.
\end{align*}
\]

We also must separate the boundary conditions:

\[
\begin{align*}
u(0, t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0, \\
u(1, t) = 0 \implies X(1)T(t) = 0 \implies X(1) = 0.
\end{align*}
\]

Thus, we obtain the following equations:

\[
\begin{align*}
\frac{d^2X}{dx^2} + \lambda X & = 0, \ X(0) = X(1) = 0, \\
\frac{dT}{dt} & = 2\lambda T = 0.
\end{align*}
\]

From our previous work, we know that the solution to the eigenvalue problem is

\[
\lambda_n = (n\pi)^2 \text{ and } X_n(x) = \sin n\pi x, \ n = 1, 2, \ldots,
\]

and

\[
T_n(t) = c_ne^{-2(n\pi)^2t}.
\]

Then, for each \( n, \ u_n(x, t) = X_n(x)T_n(t) \), so

\[
u_n(x, t) = c_ne^{-2(n\pi)^2t}\sin n\pi x.
\]
By superposition,

\[ u(x, t) = \sum_{n=1}^{\infty} c_n e^{-2(n\pi)^2 t} \sin n\pi x. \]

We need to determine the coefficients \( c_n \) using the initial condition. Since

\[ x = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin n\pi x, \quad n = 1, 2, \ldots, \]

we see that the right-hand side must be the Fourier sine series of \( x \). By Example 2 in Section 3.6, then, we have that

\[ c_n = \frac{2(-1)^{n+1}}{n\pi}. \]

Therefore,

\[ u(x, t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} e^{-2(n\pi)^2 t} \sin n\pi x. \]

2(c) Solve the heat equation \( u_t = 4u_{xx} \) for a rod of length \( L \) with both ends insulated, if \( L = 2 \),

\[ u(x, 0) = \begin{cases} 
10, & \text{if } 0 \leq x \leq 1, \\
0, & \text{if } 1 < x \leq 2.
\end{cases} \]

**Solution:** We are solving the initial-boundary-value problem

\[ u_t = 4u_{xx}, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = u(2, t) = 0, \quad t > 0, \]
\[ u(x, 0) = \begin{cases} 
10, & \text{if } 0 \leq x \leq 1, \\
0, & \text{if } 1 < x \leq 2.
\end{cases} \]

As before, we use separation of variables to solve, letting \( u(x, t) = X(x)T(t) \). Then, we obtain

\[ \frac{dT}{dt} X = 4 \frac{d^2X}{dx^2} T \]

\[ \Rightarrow 1 \frac{dT}{dt} = 1 \frac{d^2X}{X \, dx^2} = -\lambda. \]

We also must separate the boundary conditions:

\[ u_x(0, t) = 0 \implies \frac{dX}{dx}(0)T(t) = 0 \implies \frac{dX}{dx}(0) = 0, \]
\[ u_x(2, t) = 0 \implies \frac{dX}{dx}(2)T(t) = 0 \implies \frac{dX}{dx}(2) = 0. \]

Thus, we obtain the following equations:

\[ \frac{d^2X}{dx^2} + \lambda X = 0, \quad \frac{dX}{dx}(0) = \frac{dX}{dx}(2) = 0, \]
\[ \frac{dT}{dt} + 4\lambda T = 0. \]
From our previous work, we know that the solution to the eigenvalue problem is

\[ \lambda_0 = 0, \quad X_0(x) = 1, \]
\[ \lambda_n = \left( \frac{n\pi}{2} \right)^2, \quad X_n(x) = \cos \frac{n\pi x}{2}, \quad n = 1, 2, \ldots, \]

and

\[ T_0(t) = c_0, \quad T_n(t) = c_ne^{-4\left( \frac{n\pi}{2} \right)^2 t}, \quad n = 1, 2, \ldots. \]

Then, for each \( n \),

\[ u_0(x, t) = c_0 \]
\[ u_n(x, t) = c_ne^{-4\left( \frac{n\pi}{2} \right)^2 t} \cos \frac{n\pi x}{2}, \quad n = 1, 2, \ldots. \]

Therefore, by superposition,

\[ u(x, t) = c_0 + \sum_{n=1}^{\infty} c_ne^{-4\left( \frac{n\pi}{2} \right)^2 t} \cos \frac{n\pi x}{2}. \]

Next, we determine the coefficients using the initial condition. Since

\[ u(x, 0) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{2}, \]

we know that the right-hand side is the Fourier cosine series of \( u(x, 0) \), which we now must determine.

\[ 2c_0 = \frac{2}{2} \int_0^2 u(x, 0) \, dx \quad \text{(since } c_0 = \frac{a_0}{2}) \]
\[ = \int_0^1 10 \, dx + \int_1^2 0 \, dx \]
\[ = 10 \]
\[ \implies c_0 = 5. \]

\[ c_n = \frac{2}{2} \int_0^2 u(x, 0) \cos \frac{n\pi x}{2} \, dx \]
\[ = \int_0^1 10 \cos \frac{n\pi x}{2} \, dx + 0 \]
\[ = \left. \frac{20}{n\pi} \sin \frac{n\pi x}{2} \right|_0^1 \]
\[ = \frac{20}{n\pi} \sin \frac{n\pi}{2}. \]

Therefore, the solution is

\[ u(x, t) = 5 + \sum_{n=1}^{\infty} \frac{20}{n\pi} \sin \frac{n\pi x}{2} e^{-4\left( \frac{n\pi}{2} \right)^2 t} \cos \frac{n\pi x}{2}. \]
5(d) Solve the general heat equation \( u_t = \alpha^2 u_{xx} \) subject to initial conditions \( u(x,0) = f(x) \) and to the boundary conditions \( u_x(0,t) = u(L,t) = 0 \).

**Solution:** We are solving the initial-boundary-value problem

\[
\begin{align*}
  u_t &= \alpha^2 u_{xx}, \quad 0 < x < L, t > 0, \\
  u_x(0,t) &= u(L,t) = 0, \quad t > 0, \\
  u(x,0) &= f(x), \quad 0 < x < L.
\end{align*}
\]

As before, we use separation of variables to solve, letting \( u(x,t) = X(x)T(t) \). Then, we obtain

\[
\begin{align*}
  \frac{dT}{dt} X &= \alpha^2 \frac{d^2X}{dx^2} T \\
  \implies \frac{1}{\alpha^2 T} \frac{dT}{dt} &= \frac{1}{X} \frac{d^2X}{dx^2} = -\lambda.
\end{align*}
\]

We also must separate the boundary conditions:

\[
\begin{align*}
  u_x(0,t) = 0 &= \implies \frac{dX}{dx}(0)T(t) = 0 = \implies \frac{dX}{dx}(0) = 0, \\
  u(L,t) = 0 &= \implies X(L)T(t) = 0 = \implies X(L) = 0.
\end{align*}
\]

From our previous work, we know that the solution to the eigenvalue problem is

\[
\lambda_n = \left( \frac{(2n-1)\pi}{2L} \right)^2, \quad X_n(x) = \cos \left( \frac{(2n-1)\pi x}{2L} \right), \quad n = 1, 2, \ldots,
\]

and

\[
T_n(t) = c_n e^{-\alpha^2 \left( \frac{(2n-1)\pi}{2L} \right)^2 t}.
\]

Then, for each \( n \),

\[
u_n(x,t) = c_n e^{-\alpha^2 \left( \frac{(2n-1)\pi}{2L} \right)^2 t} \cos \left( \frac{(2n-1)\pi x}{2L} \right).
\]

By superposition,

\[
u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \left( \frac{(2n-1)\pi}{2L} \right)^2 t} \cos \left( \frac{(2n-1)\pi x}{2L} \right).
\]

Finally, we need to determine the coefficients \( c_n \) using the initial condition \( u(x,0) = f(x) \). We have

\[
f(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \cos \left( \frac{(2n-1)\pi x}{2L} \right).
\]

From the discussion in Section 3.7 and from Exercise 1 in that section, we know that the functions \( \left\{ \cos \left( \frac{(2n-1)\pi x}{2L} \right) \right\} \) form a complete orthogonal set on \( 0 \leq x \leq L \), and therefore, we can do the following: multiply both sides of the equation by \( \cos \left( \frac{(2m-1)\pi x}{2L} \right) \), integrate both sides with respect to \( x \), and solve. If we do this, we will obtain

\[
c_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{(2n-1)\pi x}{2L} \right) dx.
\]

Therefore, the solution is

\[
u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha^2 \left( \frac{(2n-1)\pi}{2L} \right)^2 t} \cos \left( \frac{(2n-1)\pi x}{2L} \right), \quad c_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{(2n-1)\pi x}{2L} \right) dx.
\]
2 Exercises 4.2: 2(b), 3(a), 5(a)

2(b) Solve the wave equation \( u_{tt} = 4u_{xx} \) for a string of length \( \pi \), subject to the boundary conditions \( u_x(0,t) = u_x(\pi,t) = 0 \) and to the initial conditions \( u(x,0) = \sin x, \ u_t(x,0) = 0 \). (See Exercise 3, Section 3.6.)

Solution: We are solving the initial-boundary-value problem

\[
\begin{align*}
    u_{tt} &= 4u_{xx}, & 0 < x < \pi, t > 0, \\
    u_x(0,t) &= u_x(\pi,t) = 0, & t > 0, \\
    u(x,0) &= \sin x, & 0 \leq x \leq \pi, \\
    u_t(x,0) &= 0, & 0 \leq x \leq \pi.
\end{align*}
\]

As before, we use separation of variables to solve, letting \( u(x,t) = X(x)T(t) \). Then, we obtain

\[
\frac{d^2T}{dt^2}X = 4\frac{d^2X}{dx^2}T
\]

\[
\Rightarrow \quad \frac{1}{4T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda.
\]

We also must separate the boundary conditions:

\[
\begin{align*}
    u_x(0,t) = 0 & \quad \Rightarrow \quad \frac{dX}{dx}(0)T(t) = 0 \quad \Rightarrow \quad \frac{dX}{dx}(0) = 0, \\
    u_x(\pi,t) = 0 & \quad \Rightarrow \quad \frac{dX}{dx}(\pi)T(t) = 0 \quad \Rightarrow \quad \frac{dX}{dx}(\pi) = 0.
\end{align*}
\]

Thus, we obtain the following equations:

\[
\begin{align*}
    d^2X + \lambda X &= 0, & dX(0) = dX(\pi) = 0, \\
    \frac{d^2T}{dt^2} + 4\lambda T &= 0.
\end{align*}
\]

From our previous work, we know that the solution to the eigenvalue problem is

\[
\lambda_0 = 0, \quad X_0(x) = 1, \\
\lambda_n = n^2, \quad X_n(x) = \cos nx, \quad n = 1, 2, \ldots,
\]

Then, we need to solve the equation in \( T \) for both cases of \( \lambda \).

\[
\lambda = 0 \quad \Rightarrow \quad T = c_1 + c_2t, \\
\lambda_n = n^2 \quad \Rightarrow \quad T'' + 4n^2T = 0 \\
\Rightarrow \quad T_n(t) = c_1 \cos 2nt + c_2 \sin 2nt.
\]

So, for each \( n \), \( u_n(x,t) = X_n(x)T_n(t) \), so

\[
\begin{align*}
    u_0(x,t) &= c_0 + d_0t \quad \text{and} \quad u_n(x,t) = c_n \cos 2nt \cos nx + d_n \sin 2nt \cos nx.
\end{align*}
\]

By superposition,

\[
u(x,t) = c_0 + d_0t + \sum_{n=1}^{\infty} c_n \cos 2nt \cos nx + \sum_{n=1}^{\infty} d_n \sin 2nt \cos nx.
\]
We now apply the initial conditions. First,

\[ \sin x = u(x, 0) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx. \]

Therefore, the right-hand side must be the Fourier cosine series of \( f(x) = \sin x \), which we have computed in Exercise 3.6, problem 3.

\[ F_c(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi} \cdot \frac{(1 + (-1)^n)}{n^2 - 1} \cos nx. \]

So, we see that

\[ c_0 = \frac{2}{n\pi}, \]
\[ c_n = \frac{2}{\pi} \cdot \frac{(1 + (-1)^n)}{n^2 - 1}. \]

We now have

\[ u(x, t) = \frac{2}{n\pi} + d_0 t + \sum_{n=1}^{\infty} \frac{4n}{\pi} \cdot \frac{(1 + (-1)^n)}{n^2 - 1} \sin 2nt \cos nx + \sum_{n=1}^{\infty} d_n \sin 2nt \cos nx \]

\[ \Rightarrow u_t(x, t) = d_0 + \sum_{n=1}^{\infty} -\frac{4n}{\pi} \cdot \frac{(1 + (-1)^n)}{n^2 - 1} \sin 2nt \cos nx + \sum_{n=1}^{\infty} 2nd_n \sin 2nt \cos nx \]

\[ \Rightarrow u_t(x, 0) = d_0 + \sum_{n=1}^{\infty} 2nd_n \cos nx. \]

But, since \( u_t(x, 0) = 0 \), we have

\[ d_0 = 0, \]
\[ d_n = 0, \quad n = 1, 2, \ldots. \]

Therefore, the solution is

\[ u(x, t) = \frac{2}{n\pi} + \sum_{n=1}^{\infty} \frac{2}{\pi} \cdot \frac{(1 + (-1)^n)}{n^2 - 1} \cos 2nt \cos nx. \]

3(a) Solve the wave equation \( u_{tt} = u_{xx} \) subject to \( u(0, t) = u_x(1, t), \; u(x, 0) = 0, \; u_t(x, 0) = 1. \)

**Solution:** We are solving the initial-boundary-value problem

\[ u_{tt} = u_{xx}, \quad 0 < x < 1, t > 0, \]
\[ u(0, t) = u_x(1, t) = 0, \quad t > 0, \]
\[ u(x, 0) = 0, \quad 0 \leq x \leq 1, \]
\[ u_t(x, 0) = 1, \quad 0 < x < 1. \]

As before, we use separation of variables to solve, letting \( u(x, t) = X(x)T(t) \). Then, we obtain

\[ \frac{d^2T}{dt^2} X = \frac{d^2X}{dx^2} T \]

\[ \Rightarrow \frac{1}{T} \frac{d^2T}{dt^2} = \frac{1}{X} \frac{d^2X}{dx^2} = -\lambda. \]
We also must separate the boundary conditions:
\[ u(0, t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0, \]
\[ u_x(1, t) = 0 \implies \frac{dX}{dx}(1)T(t) = 0 \implies \frac{dX}{dx}(1) = 0. \]

Thus, we obtain the following equations:
\[ \frac{d^2 X}{dx^2} + \lambda X = 0, \quad X(0) = \frac{dX}{dx}(1) = 0, \]
\[ \frac{d^2 T}{dt^2} + \lambda T = 0. \]

From previous work (Exercises 1.7, problem 16(b)), we know that the solution to the eigenvalue problem, since \( L = 1 \), is
\[ \lambda_n = \left( \frac{(2n - 1)\pi}{2} \right)^2, \quad X_n(x) = \sin \frac{(2n - 1)\pi x}{2}, \quad n = 1, 2, \ldots, \]
and therefore,
\[ T_n(t) = c_n \cos \frac{(2n - 1)\pi t}{2} + d_n \sin \frac{(2n - 1)\pi t}{2}. \]

Then, for each \( n \), \( u_n(x, t) = X_n(x)T_n(t) \), so
\[ u_n(x, t) = c_n \cos \frac{(2n - 1)\pi t}{2} \sin \frac{(2n - 1)\pi x}{2} + d_n \sin \frac{(2n - 1)\pi t}{2} \sin \frac{(2n - 1)\pi x}{2}. \]

By superposition,
\[ u(x, t) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n - 1)\pi t}{2} \sin \frac{(2n - 1)\pi x}{2} + \sum_{n=1}^{\infty} d_n \sin \frac{(2n - 1)\pi t}{2} \sin \frac{(2n - 1)\pi x}{2}. \]

We now apply the initial conditions. First,
\[ 0 = u(x, 0) = \sum_{n=1}^{\infty} c_n \sin \frac{(2n - 1)\pi x}{2} \implies c_n = 0 \text{ for all } n. \]

We, thus, have
\[ u(x, t) = \sum_{n=1}^{\infty} d_n \sin \frac{(2n - 1)\pi t}{2} \sin \frac{(2n - 1)\pi x}{2}, \]
\[ \implies u_t(x, t) = \sum_{n=1}^{\infty} \frac{(2n - 1)\pi}{2} d_n \cos \frac{(2n - 1)\pi t}{2} \sin \frac{(2n - 1)\pi x}{2} \]
\[ \implies 1 = u_t(x, 0) = \sum_{n=1}^{\infty} \frac{(2n - 1)\pi}{2} d_n \sin \frac{(2n - 1)\pi x}{2}. \]

Since \( \{ \sin \frac{(2n - 1)\pi x}{2} \} \) is a complete orthogonal set, we multiply both sides by \( \sin \frac{(2m - 1)\pi x}{2} \) and integrate with respect to \( x \) to find
\[ \int_0^1 \sin \frac{(2m - 1)\pi x}{2} dx = \sum_{n=1}^{\infty} \frac{(2n - 1)\pi}{2} d_n \int_0^1 \sin \frac{(2n - 1)\pi x}{2} \sin \frac{(2m - 1)\pi x}{2} dx, \]
and
\[ \int_0^1 \sin \left( \frac{(2m-1)\pi x}{2} \right) \, dx = -\frac{2}{(2m-1)\pi} \cos \left( \frac{(2m-1)\pi x}{2} \right) \bigg|_0^1, \]
\[ \int_0^1 \sin \left( \frac{(2n-1)\pi x}{2} \right) \sin \left( \frac{(2m-1)\pi x}{2} \right) \, dx = \begin{cases} 0, & \text{if } n \neq m, \\ \frac{1}{2}, & \text{if } n = m. \end{cases} \]

\[ \implies \frac{2}{(2m-1)\pi} = \frac{1}{4} d_m \]
\[ \implies d_m = \frac{8}{(2m-1)^2\pi^2}. \]

Therefore, we obtain the solution
\[ u(x, t) = \sum_{n=1}^{\infty} \frac{8}{(2n-1)^2\pi^2} \sin \left( \frac{(2n-1)\pi t}{2} \right) \sin \left( \frac{(2n-1)\pi x}{2} \right). \]

5(a) Solve the initial-boundary-value problem
\[ u_{tt} = u_{xx} - 4u_t, \]
\[ u(x, 0) = 1, \]
\[ u_t(x, 0) = 0, \]
\[ u(0, t) = u(\pi, t) = 0. \]

**Solution:** We will use separation of variables to solve this problem. Let \( u(x, t) = X(x)T(t) \). Then
\[ u_t = X(x) \frac{dT}{dt}, \quad u_{tt} = X(x) \frac{d^2T}{dt^2}, \quad u_{xx} = \frac{d^2X}{dx^2}. \]

Substituting into the PDE gives
\[ X(x) \frac{d^2T}{dt^2} = T(t) \frac{d^2X}{dx^2} - 4X(x) \frac{dT}{dt}, \]
\[ X(x) \frac{d^2T}{dt^2} + 4X(x) \frac{dT}{dt} = T(t) \frac{d^2X}{dx^2}. \]

Divide both sides by \( X(x)T(t) \) to obtain
\[ \frac{1}{T} \left( \frac{d^2T}{dt^2} + 4 \frac{dT}{dt} \right) = \frac{1}{X} \frac{d^2X}{dx^2} = -\lambda. \]

We also separate the boundary conditions.
\[ u(0, t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0 \]
\[ u(\pi, t) = 0 \implies X(\pi)T(t) = 0 \implies X(\pi) = 0. \]

Thus, we obtain the following equations:
\[ X'' + \lambda X = 0, \quad X(0) = X(\pi) = 0, \]
\[ \frac{d^2T}{dt^2} + 4 \frac{dT}{dt} + \lambda T = 0. \]
From our previous work, we know that the solution to the eigenvalue problem is
\[ \lambda_n = n^2, \quad X_n(x) = \sin nx, \quad n = 1, 2, \ldots. \]

Then, we must solve the equation in \( T \):
\[ T'' + 4T' + n^2 T = 0 \]

Characteristic equation:
\[ r^2 + 4r + n^2 = 0 \]
\[ r_{1,2} = \frac{-4 \pm \sqrt{16 - 4n^2}}{2} \]
\[ = -2 \pm \sqrt{4 - n^2} \]

If \( n = 1 \), then
\[ r_{1,2} = -2 \pm \sqrt{3} \]
\[ \implies T_1(t) = c_1 e^{-2t} \cosh \sqrt{3}t + d_1 e^{-2t} \sinh \sqrt{3}t. \]

If \( n = 2 \), then
\[ r_1 = r_2 = -2 \]
\[ \implies T_2(t) = c_2 e^{-2t} + d_2 t e^{-2t}. \]

If \( n > 2 \), then
\[ r_{1,2} = -2 \pm i \sqrt{n^2 - 4} \]
\[ \implies T_n(t) = c_n e^{-2t} \cos(t \sqrt{n^2 - 4}) + d_n e^{-2t} \sin(t \sqrt{n^2 - 4}). \]

This gives the following, then, for \( u_n(x, t) \)
\[ u_1(x, t) = c_1 e^{-2t} \cosh \sqrt{3}t \sin x + d_1 e^{-2t} \sinh \sqrt{3}t \sin x, \]
\[ u_2(x, t) = c_2 e^{-2t} \sin 2x + d_2 t e^{-2t} \sin 2x, \]
\[ u_n(x, t) = c_n e^{-2t} \cos(t \sqrt{n^2 - 4}) \sin nx + d_n e^{-2t} \sin(t \sqrt{n^2 - 4}) \sin nx, \quad n = 3, 4, \ldots. \]

By superposition, we have
\[ u(x, t) = c_1 e^{-2t} \cosh \sqrt{3}t \sin x + d_1 e^{-2t} \sinh \sqrt{3}t \sin x + c_2 e^{-2t} \sin 2x + d_2 t e^{-2t} \sin 2x \]
\[ + \sum_{n=3}^{\infty} c_n e^{-2t} \cos(t \sqrt{n^2 - 4}) \sin nx + \sum_{n=3}^{\infty} d_n e^{-2t} \sin(t \sqrt{n^2 - 4}) \sin nx. \]

We now apply the initial conditions. First,
\[ 1 = u(x, 0) = c_1 \sin x + c_2 \sin 2x + \sum_{n=3}^{\infty} c_n \sin nx \]
\[ = \sum_{n=1}^{\infty} c_n \sin nx. \]
Therefore, the right-hand side is the Fourier sine series of \( u(x, 0) = 1 \), and so

\[
c_n = b_n = \frac{2}{\pi} \int_0^\pi \sin n x \, dx
\]

\[
= -\frac{2}{n \pi} \cos n \pi x \bigg|_0^\pi
\]

\[
= \frac{2}{n \pi} (1 - \cos n \pi)
\]

\[
= \frac{2}{n \pi} (1 - (-1)^n).
\]

So, we have

\[
u(x, t) = \frac{4}{\pi} e^{-2t} \cosh \sqrt{3}t \sin x + d_1 e^{-2t} \sinh \sqrt{3}t \sin x + d_2 t e^{-2t} \sin 2x
\]

\[
+ \sum_{n=3}^\infty \frac{2}{n \pi} (1 - (-1)^n) e^{-2t} \cos(t \sqrt{n^2 - 4}) \sin n x + \sum_{n=3}^\infty d_n e^{-2t} (\sin t \sqrt{n^2 - 4}) \sin n x
\]

\[
\implies u_t(x, t) = -\frac{8}{\pi} e^{-2t} \cosh \sqrt{3}t \sin x + \frac{4 \sqrt{3}}{\pi} e^{-2t} \sinh \sqrt{3}t \sin x - 2d_1 e^{-2t} \sinh \sqrt{3}t \sin x
\]

\[
+ \sqrt{3} d_1 e^{-2t} \cosh \sqrt{3}t \sin x + d_2 e^{-2t} \sin 2x - 2d_2 t e^{-2t} \sin 2x
\]

\[
+ \sum_{n=3}^\infty \frac{-4}{n \pi} e^{-2t} \cos(t \sqrt{n^2 - 4}) \sin n x + \sum_{n=3}^\infty \frac{2 \sqrt{n^2 - 4}}{n \pi} (1 - (-1)^n) \sin(t \sqrt{n^2 - 4}) \sin n x
\]

\[
+ \sum_{n=3}^\infty -2d_n e^{-2t} \sin(t \sqrt{n^2 - 4}) \sin n x + \sum_{n=3}^\infty -d_n \sqrt{n^2 - 4} e^{-2t} \cos(t \sqrt{n^2 - 4}) \sin n x.
\]

Therefore,

\[
0 = u_t(x, 0) = -\frac{8}{\pi} \sin x + \sqrt{3} d_1 \sin x + d_2 \sin 2x + \sum_{n=3}^\infty \frac{-4}{n \pi} \sin n x + \sum_{n=3}^\infty -d_n \sqrt{n^2 - 4} \sin n x.
\]

Re-arranging gives

\[
\implies \frac{8}{\pi} \sin x + \sum_{n=3}^\infty \frac{4}{n \pi} \sin n x = \sqrt{3} d_1 \sin x + d_2 \sin 2x + \sum_{n=3}^\infty -d_n \sqrt{n^2 - 4} \sin n x.
\]

Equating coefficients, we see that:

\[
\sqrt{3} d_1 = \frac{8}{\pi} \implies d_1 = \frac{8}{\sqrt{3} \pi}
\]

\[
d_2 = 0
\]

\[
-\sqrt{n^2 - 4} d_n = \frac{4}{n \pi}, \quad n \geq 3,
\]

\[
\implies d_n = -\frac{4}{\pi n \sqrt{n^2 - 4}}, \quad n \geq 3.
\]

Therefore, the solution is

\[
u(x, t) = \frac{4}{\pi} e^{-2t} \cosh \sqrt{3}t \sin x + \frac{8}{\sqrt{3} \pi} e^{-2t} \sinh \sqrt{3}t \sin x + \sum_{n=3}^\infty \frac{2}{n \pi} (1 - (-1)^n) e^{-2t} \cos(t \sqrt{n^2 - 4}) \sin n x
\]

\[
+ \sum_{n=3}^\infty -\frac{4}{\pi n \sqrt{n^2 - 4}} e^{-2t} \sin(t \sqrt{n^2 - 4}) \sin n x.
\]
Since \( \lim_{t \to \infty} e^{-2t} = 0 \), \( |\cos at| \leq 1 \), \( |\sin at| \leq 1 \), and \( \sqrt{3} < 2 \), we have \( \lim_{t \to \infty} u(x,t) = 0 \).

3 Exercises 4.3: 3, 7, 12(c)

For problems 3 and 7, solve Laplace’s equation subject to the given boundary conditions. Work each problem out completely, rather than referring to the solutions in this section.

3. \( u(x,0) = u(x, \pi) = 0 \), \( u(0,y) = y \), \( u(1,y) = 0 \)

**Solution:** We are solving the initial-boundary-value problem

\[
\triangle u = 0, \quad 0 < x < 1, 0 < y < \pi, \\
u(x,0) = u(x, \pi) = 0, \quad 0 < x < 1, \\
u(0,y) = y, \quad 0 < y < \pi, \\
u(1,y) = 0, \quad 0 < y < \pi.
\]

We will solve this problem using separation of variables. Let \( u(x,y) = X(x)Y(y) \). Then, we obtain

\[
\frac{d^2 X}{dx^2} Y(y) + \frac{d^2 Y}{dy^2} X(x) = 0.
\]

Divide by \( XY \) to obtain

\[
\frac{1}{X} \frac{d^2 X}{dx^2} = - \frac{1}{Y} \frac{d^2 Y}{dy^2} = \lambda.
\]

We need to obtain the boundary conditions, as well:

\[
0 = u(x,0) = X(x)Y(0) \implies Y(0) = 0, \\
0 = u(x, \pi) = X(x)Y(\pi) \implies Y(\pi) = 0.
\]

We thus obtain the following equations:

\[
Y'' + \lambda Y = 0, \quad Y'(0) = Y'(\pi) = 0, \\
X'' - \lambda X = 0.
\]

The eigenvalue problem is similar to one we have seen before, so we know that

\[
\lambda_n = n^2, \quad Y_n(y) = \sin ny, \quad n = 1, 2, \ldots.
\]

We then solve the equation for \( X \):

\[
X'' - n^2 X = 0
\]

The characteristic equation is

\[
r^2 - n^2 = 0 \\
\implies r_{1,2} = \pm n \\
\implies X_n(x) = c_n \cosh nx + d_n \sinh nx.
\]

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For each \( n \), \( u_n(x, y) = X_n(x)Y_n(y) \) is given by
\[
u_n(x, y) = c_n \cosh nx \sin ny + d_n \sinh nx \sin ny.
\]

By superposition,
\[
u(x, y) = \sum_{n=1}^{\infty} c_n \cosh nx \sin ny + \sum_{n=1}^{\infty} d_n \sinh nx \sin ny.
\]

We now may apply the remaining boundary conditions to determine \( c_n \) and \( d_n \).
\[
y = u(0, y) = \sum_{n=1}^{\infty} c_n \sin ny.
\]

Therefore, the right-hand side represents the Fourier sine series of \( u(0, y) = y \). This tells us
\[
c_n = \frac{2}{\pi} \int_0^{\pi} y \sin ny \, dy = \frac{2}{n} \cos n\pi = \frac{2}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}.
\]

So, we have
\[
u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \cosh nx \sin ny + \sum_{n=1}^{\infty} d_n \sinh nx \sin ny
\]
\[
\implies 0 = u(1, y) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \cosh n \sin ny + \sum_{n=1}^{\infty} d_n \sinh n \sin ny
\]
\[
\implies \sum_{n=1}^{\infty} d_n \sinh n \sin ny = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \cosh n \sin ny.
\]

Equating coefficients requires
\[
d_n \sinh n = \frac{2}{n} (-1)^{n+1} \cosh n
\]
\[
\implies d_n = \frac{2}{n} (-1)^{n+1} \coth n.
\]

Therefore, the solution to our problem is
\[
u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \cosh nx \sin ny + \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \coth n \sinh nx \sin ny.
\]

7. \( u_y(x, 0) = 0, \; u(x, 1) = x, \; u(0, y) = u(1, y) = 0 \)

**Solution:** We are solving the initial-boundary-value problem
\[
\Delta u = 0, \quad 0 < x < 1, 0 < y < 1,
u_y(x, 0) = 0, \quad 0 < x < 1,
u_y(x, 1) = x, \quad 0 < x < 1,
u(0, y) = u(1, y) = 0, \quad 0 < y < 1.
\]
We will solve this problem using separation of variables. Let \( u(x, y) = X(x)Y(y) \). Then, we obtain
\[
\frac{d^2X}{dx^2}Y(y) + \frac{d^2Y}{dy^2}X(x) = 0.
\]
Divide by \( XY \) to obtain
\[
\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{d^2Y}{dy^2} = -\lambda.
\]
We need to obtain the boundary conditions, as well:
\[
0 = u(0, y) = X(0)Y(y) \implies X(0) = 0,
0 = u(1, y) = X(1)Y(y) \implies X(1) = 0.
\]
We thus obtain the following equations:
\[
X'' + \lambda X = 0, \quad X(0) = X(1) = 0,
Y'' - \lambda Y = 0.
\]
The eigenvalue problem is similar to one we have seen before, so we know that
\[
\lambda_n = (n\pi)^2, \quad X_n(x) = \sin n\pi x, \quad n = 1, 2, \ldots.
\]
We then solve the equation for \( X \):
\[
Y'' - (n\pi)^2 Y = 0
\]
The characteristic equation is
\[
r^2 - (n\pi)^2 = 0
\]
\[
\implies r_{1,2} = \pm n\pi
\]
\[
\implies Y_n(y) = c_n \cosh n\pi y + d_n \sinh n\pi y.
\]
For each \( n \), \( u_n(x, y) = X_n(x)Y_n(y) \) is given by
\[
u_n(x, y) = c_n \cosh n\pi y \sin n\pi x + d_n \sinh n\pi y \sin n\pi x.
\]
By superposition,
\[
u(x, y) = \sum_{n=1}^{\infty} c_n \cosh n\pi y \sin n\pi x + \sum_{n=1}^{\infty} d_n \sinh n\pi y \sin n\pi x.
\]
We now may apply the remaining boundary conditions to determine \( c_n \) and \( d_n \).
\[
0 = u_y(x, 0) = \sum_{n=1}^{\infty} d_n n\pi \cos n\pi x,
\]
\[
\implies d_n = 0, \quad n = 1, 2, \ldots
\]
\[
x = u(x, 1) = \sum_{n=1}^{\infty} c_n \cosh n\pi \sin n\pi x.
\]
Therefore, the right-hand side must represent the Fourier sine series for \( u(x, 1) = x \).

\[
F_s(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi x
\]

\[
\Rightarrow c_n \cosh n\pi = \frac{2(-1)^{n+1}}{n\pi} \cosh n\pi
\]

\[
\Rightarrow c_n = \frac{2}{n\pi \cosh n\pi} (-1)^{n+1}.
\]

Therefore, the solution to the problem is

\[
u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n\pi \cosh n\pi} (-1)^{n+1} \cosh n\pi y \sin n\pi x.
\]

12(c) What is the solution of the Neumann problem

\[
u_{xx} + \nu_{yy} = 0,
\]

\[
u_y(x, 0) = f_1(x),
\]

\[
u_y(x, b) = f_2(x),
\]

\[
u_x(0, y) = g_1(y),
\]

\[
u_x(a, y) = g_2(y).
\]

**Solution:** First, write \( \nu(x, y) = \nu_1(x, y) + \nu_2(x, y) \), where \( \nu_1 \) solves

\[
u_{1xx} + \nu_{1yy} = 0,
\]

\[
u_{1y}(x, 0) = 0,
\]

\[
u_{1y}(x, b) = 0,
\]

\[
u_{1x}(0, y) = g_1(y),
\]

\[
u_{1x}(a, y) = g_2(y),
\]

and \( \nu_2 \) solves

\[
u_{2xx} + \nu_{2yy} = 0,
\]

\[
u_{2y}(x, 0) = f_1(x),
\]

\[
u_{2y}(x, b) = f_2(x),
\]

\[
u_{2x}(0, y) = 0,
\]

\[
u_{2x}(a, y) = 0.
\]

First, solve for \( \nu_1(x, y) \). Write \( \nu_1(x, y) = X_1(x)Y_1(y) \). Then, we have

\[
Y_1(y) \frac{d^2X_1}{dx^2} + X_1(x) \frac{d^2Y_1}{dy^2} = 0.
\]

Divide both sides by \( X_1Y_1 \) to obtain

\[
\frac{1}{X_1(x)} \frac{d^2X_1}{dx^2} = \frac{1}{Y_1(x)} \frac{d^2Y_1}{dy^2} = \lambda.
\]
We need to obtain the boundary conditions, as well.

\[ 0 = u_{1_y}(x, 0) = X_1(x)Y_1(0) \implies Y_1(0) = 0, \]
\[ 0 = u_{1_y}(x, b) = X_1(x)Y_1(b) \implies Y_1(b) = 0. \]

We thus obtain the following equations:

\[ Y_1'' + \lambda Y_1 = 0, \quad Y_1'(0) = Y_1'(b) = 0, \]
\[ X_1'' - \lambda X_1 = 0. \]

The eigenvalue problem is similar to one we have seen before, so we know that

\[ \lambda_0 = 0, \quad Y_{1_0}(y) = 1, \]
\[ \lambda_n = \left( \frac{n\pi}{b} \right)^2, \quad Y_{1_n}(y) = \cos \frac{n\pi y}{b}, \quad n = 1, 2, \ldots. \]

Then, solving for \( X_1 \) gives us

\[ X_{1_0}(x) = c_0 + d_0 x, \]
\[ X_{1_n}(x) = c_n \cosh \frac{n\pi x}{b} + d_n \sinh \frac{n\pi x}{b}. \]

For each \( n \), then \( u_{1_n}(x, y) \) is given by

\[ u_{1_0}(x, y) = c_0 + d_0 x, \]
\[ u_{1_n}(x, y) = c_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b} + d_n \sinh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}, \quad n = 1, 2, \ldots. \]

By superposition, then

\[ u_1(x, y) = c_0 + d_0 x + \sum_{n=1}^{\infty} c_n \cosh \frac{n\pi x}{b} \cos \frac{n\pi y}{b} + \sum_{n=1}^{\infty} d_n \sinh \frac{n\pi x}{b} \cos \frac{n\pi y}{b}. \]

We must apply the boundary conditions to solve for \( c_n \) and \( d_n \).

\[ g_1(y) = u_{1_x}(0, y) = c_0 + \sum_{n=1}^{\infty} \cos \frac{n\pi y}{b}. \]

Therefore, the series on the right-hand side must be the Fourier cosine series of \( g_1(y) \), and

\[ 2c_0 = \frac{2}{b} \int_0^b g_1(y) \, dy \implies c_0 = \frac{1}{b} \int_0^b g_1(y) \, dy, \]
\[ c_n = \frac{2}{b} \int_0^b g_1(y) \cos \frac{n\pi y}{b} \, dy. \]

\[ g_2(y) = u_{1_x}(a, y) = c_0 + d_0 a + \sum_{n=1}^{\infty} c_n \cosh \frac{n\pi a}{b} \cos \frac{n\pi y}{b} + \sum_{n=1}^{\infty} d_n \sinh \frac{n\pi a}{b} \cos \frac{n\pi y}{b} \]
\[ \implies g_2(y) = c_0 + d_0 a + \sum_{n=1}^{\infty} \left( c_n \cosh \frac{n\pi a}{b} + d_n \sinh \frac{n\pi a}{b} \right) \cos \frac{n\pi y}{b}. \]
The right-hand side must represent the Fourier cosine series of \( g_2(y) \), and so
\[
2(c_0 + d_0a) = \frac{2}{b} \int_0^b g_2(y) \, dy \quad \Rightarrow \quad d_0 = \frac{1}{a} \left( \frac{1}{b} \int_0^b g_2(y) \, dy - c_0 \right)
\]
\[
c_n \cosh \frac{n\pi a}{b} + d_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b g_2(y) \cos \frac{n\pi y}{b} \, dy
\]
\[
\Rightarrow d_n = \frac{1}{\sinh \frac{n\pi a}{b}} \left( \frac{2}{b} \int_0^b g_2(y) \cos \frac{n\pi y}{b} \, dy - c_n \cosh \frac{n\pi a}{b} \right).
\]
Therefore, the solution is given by
\[
u_1(x, y) = c_0 + d_0x + \sum_{n=1}^\infty c_n \cosh \frac{n\pi a}{b} x \cos \frac{n\pi y}{b} + \sum_{n=1}^\infty d_n \sinh \frac{n\pi a}{b} x \cos \frac{n\pi y}{b},
\]
where \( c_n \) and \( d_n \), \( n = 0, 1, \ldots \) are defined as above. Next, we solve for \( u_2(x, y) \) using separation of variables. Let \( u_2(x, y) = X_2(x)Y_2(y) \). Then,
\[
Y_2(y) \frac{d^2X_2}{dx^2} + X_2(x) \frac{d^2Y_2}{dy^2} = 0.
\]
Divide both sides by \( X_2Y_2 \) to obtain
\[
\frac{1}{X_2(x)} \frac{d^2X_2}{dx^2} = -\frac{1}{Y_2(x)} \frac{d^2Y_2}{dy^2} = -\lambda.
\]
We need to obtain the boundary conditions, as well.
\[
0 = u_1_y(0, y) = X_2_x(0)Y_2(y) \quad \Rightarrow \quad X_2_x(0) = 0,
\]
\[
0 = u_1_y(a, y) = X_2_x(a)Y_2(y) \quad \Rightarrow \quad X_2_x(a) = 0.
\]
We thus obtain the following equations:
\[
X_2'' + \lambda X_2 = 0 \quad X_2'(0) = X_2'(a) = 0,
\]
\[
Y_2'' - \lambda Y_2 = 0.
\]
The eigenvalue problem is similar to one we have seen before, so we know that
\[
\lambda_0 = 0, \quad X_2_0(x) = 1,
\]
\[
\lambda_n = \left( \frac{n\pi a}{a} \right)^2, \quad X_2_n(y) = \cos \frac{n\pi x}{a}, \quad n = 1, 2, \ldots.
\]
Then, solving for \( Y_2 \) gives us
\[
Y_2_0(y) = p_0 + q_0 y,
\]
\[
Y_2_n(y) = p_n \cosh \frac{n\pi y}{a} + q_n \sinh \frac{n\pi y}{a}.
\]
For each \( n \), then \( u_2_n(x, y) \) is given by
\[
u_2_0(x, y) = p_0 + q_0 y,
\]
\[
u_2_n(x, y) = p_n \cosh \frac{n\pi y}{a} \cos \frac{n\pi x}{a} + q_n \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a},
\]

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and by superposition,
\[ u_2(x, y) = p_0 + q_0 y + \sum_{n=1}^{\infty} p_n \cosh \frac{n\pi y}{a} \cos \frac{n\pi x}{a} + \sum_{n=1}^{\infty} q_n \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}. \]

We must apply the boundary conditions to solve for \( p_n \) and \( q_n \).

\[ f_1(x) = p_0 + \sum_{n=1}^{\infty} p_n \cos \frac{n\pi x}{a} \]

Therefore, the right-hand side is the Fourier cosine series of \( f_1(x) \). So,
\[ 2p_0 = \frac{2}{a} \int_0^a f_1(x) \, dx \quad \Rightarrow \quad p_0 = \frac{1}{a} \int_0^a f_1(x) \, dx \]
\[ p_n = \frac{2}{a} \int_0^a f_1(x) \cos \frac{n\pi x}{a} \, dx. \]

\[ f_2(x) = u_2(x, b) = p_0 + q_0 b + \sum_{n=1}^{\infty} p_n \cosh \frac{n\pi b}{a} \cos \frac{n\pi x}{a} + \sum_{n=1}^{\infty} q_n \sinh \frac{n\pi b}{a} \cos \frac{n\pi x}{a} \]
\[ \Rightarrow g_2(y) = p_0 + q_0 b + \sum_{n=1}^{\infty} \left( p_n \cosh \frac{n\pi b}{a} + q_n \sinh \frac{n\pi b}{a} \right) \cos \frac{n\pi x}{a}. \]

The right-hand side must represent the Fourier cosine series of \( f_2(x) \), and so
\[ 2(p_0 + q_0 b) = \frac{2}{a} \int_0^a f_2(x) \, dx \quad \Rightarrow \quad q_0 = \frac{1}{b} \left( \frac{1}{a} \int_0^a f_2(x) \, dx - p_0 \right) \]
\[ p_n \cosh \frac{n\pi b}{a} + q_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f_2(x) \cos \frac{n\pi x}{a} \, dx \]
\[ \Rightarrow q_n = \frac{1}{\sinh \frac{n\pi b}{a}} \left( \frac{2}{a} \int_0^a f_2(x) \cos \frac{n\pi x}{a} \, dx - p_n \cosh \frac{n\pi b}{a} \right). \]

Therefore, the solution is given by
\[ u_2(x, y) = p_0 + q_0 y + \sum_{n=1}^{\infty} p_n \cosh \frac{n\pi y}{a} \cos \frac{n\pi x}{a} + \sum_{n=1}^{\infty} q_n \sinh \frac{n\pi y}{a} \cos \frac{n\pi x}{a}, \]

where \( p_n \) and \( q_n, n = 0, 1, \ldots \) are defined as above. Then,
\[ u(x, y) = u_1(x, y) + u_2(x, y). \]