Chapter 6 is called, “Applications of Integration.” Now that we have some decent tools to evaluate integrals, we are going to explore some of the useful things we can calculate using them. The first important application is volumes of solids; the other main one is work (in the physics sense).

In §6.1 we study how to find the area between two curves, which, as it turns out, will be crucial in order to calculate volumes of solids.

Recall. The area under $f(x)$ from $x = a$ to $x = b$ is $\int_a^b f(x) \, dx$. When we say “the area under $f(x)$” we really mean “the area between $f(x)$ and the $x$-axis.” How would we find the area between two arbitrary (continuous) functions?

Example. What is the area between $f(x)$ and $g(x)$, as shown ($f(x)$ is the one on the top), between $x = 1$ and $x = 2.5$?

![Figure 1: Area between two curves](image)

Answer. Area under $f(x) - $ Area under $g(x)$

$$= \int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

$$= \int_a^b (f(x) - g(x)) \, dx.$$ 

Notice that the “curve on top”, $f(x)$ in this case, came first.

Example. Find the area between $f(x) = x^3$ and $g(x) = x^2$ from 0 to 1.

Solution. Draw graphs of $f(x) = x^3$ and $g(x) = x^2$ from 0 to 1:
The graph of __________is on top between 0 and 1. Therefore the area is

What if the function which is on top changes from $a$ to $b$?

Figure 2: $f(x)$ and $g(x)$ “change places”

**Answer.** Do it in pieces: Area $= \int_{1}^{c} g(x) - f(x) \, dx + \int_{c}^{5} f(x) - g(x) \, dx$. You have to figure out what the value of $c$ is!

**Example.** Find the area between $y = x^3$ and $y = \frac{1}{2}(x + 1)$ from 0 to 2.

**Solution.** Draw graphs of $y = x^3$ and $y = \frac{1}{2}(x + 1)$ from 0 to 2:

The graph of __________is on top at 0. But __________is on top at 2. So they must have crossed!

Find the point(s) of intersection: set $\frac{1}{2}(x + 1) = x^3$.

$x^3 - \frac{1}{2}x - \frac{1}{2} = 0$.

Remember the Intermediate Value Theorem? Use it to locate a root, or use *Mathematica* to graph both functions to see where the two functions intersect, then try a few values.

From the picture (Figure 3), it looks close to $x = 1$. Sure enough, $1^3 - \frac{1}{2}(1) - \frac{1}{2} = 0$. So the area is
Other language for finding area between curves: “Find the area enclosed by the curves . . .”.

**Tactic.** Draw a graph to see which region they’re actually talking about (where the curves are likely to intersect, which function is on top and when, etc.), compute the exact intersection points, if possible, and compute the integral.

**Example.** Find the area enclosed by $y = x^2$ and $y = x^3$.

Recall that $x^2$ and $x^3$ intersect at 0 and 1. Do they intersect anywhere else?

Setting $x^2 = x^3$, we see that $x^2(x - 1) = 0$. Therefore the only solutions are $x = 0$ and $x = 1$.

So the area we are being asked to find is the same one we found previously.