1. Proposition 2.18(iii): For all $k \in \mathbb{N}$, $k^3 + 5k$ is divisible by 6.

When trying to prove this using induction, we get stuck on the following: is $k(k+1)$ even for all $k \in \mathbb{N}$? So first, we prove this. Then we prove the actual exercise.

**Lemma:** For all $k \in \mathbb{N}$, $k(k+1)$ is even.

**Proof:** We use induction on $k$. Note that $P(k)$ is the statement ‘$k(k+1)$ is even’.

For $k = 1$: $k(k+1) = 1 \cdot (1+1) = 2$ is even. So $P(1)$ is true.

Now assume that $P(n)$ is true for some $n \in \mathbb{N}$. Thus $n(n+1)$ is even. Then

$$(n+1)((n+1) + 1) = (n+1)(n+2) = (n+1)n + (n+1)2 = n(n+1) + 2(n+1)$$

By the induction hypothesis, we have that $n(n+1)$ is even. So $n(n+1) = 2m$ for some $m \in \mathbb{Z}$. Hence

$$(n+1)(n+2) = n(n+1) + 2(n+1) = 2m + 2(n+1) = 2(m + n + 1)$$

Since $m + n + 1 \in \mathbb{Z}$, we have that $(n+1)(n+2)$ is even. So $P(n+1)$ is true. $\square$

Now we can prove Proposition 2.18(iii). Here we have that $P(k)$ is the statement ‘$k^3 + 5k$ is divisible by 6’.

For $k = 1$: $k^3 + 5k = 1^3 + 5 \cdot 1 = 6 = 6 \cdot 1$ is divisible by 6. So $P(1)$ is true.

Now assume that $P(n)$ is true for some $n \in \mathbb{N}$. Thus $n^3 + 5n$ is divisible by 6. Then

$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + (3n^2 + 3n + 6) = (n^3 + 5n) + 3n(n+1) + 6$$

By the induction hypothesis, we have that $n^3 + 5n$ is divisible by 6. So $n^3 + 5n = 6m$ for some $m \in \mathbb{Z}$. By the Lemma, we have that $n(n+1)$ is even. So $n(n+1) = 2s$ for some $s \in \mathbb{Z}$. Hence

$$(n+1)^3 + 5(n+1) = (n^3 + 5n) + 3n(n+1) + 6 = 6m + 3 \cdot 2s + 6 = 6m + 6s + 6 = 6(m + s + 1)$$

Since $m + s + 1 \in \mathbb{Z}$, we have that $(n+1)^3 + 5(n+1)$ is divisible by 6. So $P(n+1)$ is true. $\square$

2. Proposition 2.20: For all $k \in \mathbb{N}$, $k \geq 1$.

**Proof:** We use induction on $k$. Note that $P(k)$ is the statement ‘$k \geq 1$’.

For $k = 1$: Since $1 = 1$, we have that $1 \geq 1$. So $P(1)$ is true.

Now assume that $P(n)$ is true for some $n \in \mathbb{N}$. Thus $n \geq 1$. Then $n+1 \geq 1+1$. So $n+1 \geq 2$. Since $2 - 1 = 1 \in \mathbb{N}$, we have that $2 > 1$. Thus $n+1 \geq 2$ and $2 > 1$. It follows that $n+1 > 1$. Hence $n+1 \geq 1$ and $P(n+1)$ is true. $\square$
3. Proposition 2.21: There exists no integer $x$ such that $0 < x < 1$.

\textit{Proof}: Suppose that there exists an integer $x$ such that $0 < x < 1$. Then $0 < x$ and $x < 1$. Since $0 < x$, we get that $x - 0 \in \mathbb{N}$. So $x \in \mathbb{N}$. By Proposition 2.20, we have that $x \geq 1$. But $x < 1$, a contradiction to Proposition 2.8.

Hence there exists no integer $x$ such that $0 < x < 1$. \hfill $\Box$

4. Proposition 2.27: For all integers $k \geq 2$, $k^2 < k^3$.

\textit{Proof}: We use induction on $k$. Note that $P(k)$ is the statement '$k^2 < k^3$'.

For $k = 2$: Then $k^2 = 2^2 = 4 < 8 = 2^3 = k^3$. So $k^2 < k^3$ and $P(2)$ is true.

Now assume that $P(n)$ is true for some $n \in \mathbb{N}$ with $n \geq 2$. Thus $n^2 < n^3$. Note that $(n+1)^2 = n^2 + 2n + 1$. By the induction hypothesis, we have that $n^2 < n^3$.

Adding $2n + 1$, to both sides, we get

$$n^2 + 2n + 1 < n^3 + 2n + 1$$

So

$$(n+1)^2 < n^3 + 2n + 1$$

We want to show that $n^3 + 2n + 1 \leq (n+1)^3$. Note that

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1 = (n^3 + 2n + 1) + (3n^2 + n + 1)$$

Since $n > 0$, we easily get that $0 < 3n^2 + n + 1$. Adding $n^3 + 2n + 1$ to both sides, we get

$$n^3 + 2n + 1 < (n^3 + 2n + 1) + (3n^2 + n + 1)$$

Hence

$$n^3 + 2n + 1 < (n+1)^3$$

Since we know that $(n+1)^2 < n^3 + 2n + 1$ and $n^3 + 2n + 1 < (n+1)^3$, we get that

$$(n+1)^2 < (n+1)^3$$

So $P(n+1)$ is true. \hfill $\Box$

\textbf{Remark}: We can easily prove this proposition without induction. Let $k \in \mathbb{N}$ with $k \geq 2$. Since $2 > 1$, we get that $k > 1$. Multiplying both sides by $k^2$ (which is okay since $k^2 \in \mathbb{N}$), we get $k \cdot k^2 > 1 \cdot k^2$. So $k^3 > k^2$.

5. Similar to project 2.28: Determine for which natural numbers $k$ we have that $2^k > 3k - 1$. Then prove your answer.

\textit{Solution}: For $k = 1$: $2^1 = 2$ and $3 \cdot 1 - 1 = 2$ so $2^k > 3k - 1$ is false.

For $k = 2$: $2^2 = 4$ and $3 \cdot 2 - 1 = 5$ so $2^k > 3k - 1$ is false.

For $k = 3$: $2^3 = 8$ and $3 \cdot 3 - 1 = 8$ so $2^k > 3k - 1$ is false.

For $k = 4$: $2^4 = 16$ and $3 \cdot 4 - 1 = 11$ so $2^k > 3k - 1$ is true.

For $k = 5$: $2^5 = 32$ and $3 \cdot 5 - 1 = 14$ so $2^k > 3k - 1$ is true.

For $k = 6$: $2^6 = 64$ and $3 \cdot 6 - 1 = 17$ so $2^k > 3k - 1$ is true.

Hence we claim that
For all integers $k \geq 5$, $2^k > 3k - 1$.

We use induction on $k$. Note that $P(k)$ is the statement ‘$2^k > 3k - 1$’.

For $k = 5$: Then $2^5 = 32$ and $3 \cdot 5 - 1 = 14$. So $2^k > 3k - 1$ and $P(5)$ is true.

Now assume that $P(n)$ is true for some $n \in \mathbb{N}$ with $n \geq 5$. Thus $2^n > 3n - 1$. Note that $2^{n+1} = 2 \cdot 2^n$. By the induction hypothesis, we have that $2^n > 3n - 1$

Multiplying both sides by 2, we get $2 \cdot 2^n > 2(3n - 1)$

So

$$2^{n+1} > 6n - 2$$

We want to show that $6n - 2 \geq 3(n + 1) - 1$. Note that $3(n + 1) - 1 = 3n + 2$ and $6n - 2 = (3n + 2) + (3n - 4)$. Since $n \geq 5$, we get that $3n \geq 3 \cdot 5$. So $3n \geq 15$. Hence $3n - 4 \geq 15 - 4$. So $3n - 4 \geq 11$. Clearly, $11 > 0$. Thus $3n - 4 > 0$. Adding $3n + 2$ to both sides, we find that

$$(3n - 4) + (3n + 2) > 0 + (3n + 2)$$

Hence

$$6n - 2 > 3n + 2$$

Since $2^{n+1} > 6n - 2$ and $6n - 2 > 3n + 2$, we get that $2^{n+1} > 3n + 2$

So

$$2^{n+1} > 3(n + 1) - 1$$

Thus $P(n + 1)$ is true. \qed

6. Project 2.35: Compute gcd(7, 13) and gcd(−5, 15). Justify/prove your answer!

Solution: Recall that gcd(7, 13) is the smallest element of the set

$$S := \{k \in \mathbb{N} : k = 7x + 13y \text{ for some } x, y \in \mathbb{Z}\}$$

Note that $1 = 7 \cdot 2 = 13 \cdot (-1)$. So $1 \in S$. Since $1 \in S \subseteq \mathbb{N}$ and 1 is the smallest element of $\mathbb{N}$, we must have that 1 is the smallest element in $S$. Thus $\text{gcd}(7, 13) = 1$.

Recall that gcd(−5, 15) is the smallest element of the set

$$S := \{k \in \mathbb{N} : k = -5x + 15y \text{ for some } x, y \in \mathbb{Z}\}$$

Note that $5 = (-5) \cdot (-1) + 15 \cdot 0$. So $5 \in S$. We show that 5 is the smallest element in $S$. Let $k$ be an element in $S$. Then $k = -5x + 15y$ for some $x, y \in \mathbb{Z}$. So $k = 5(-x + 3y)$. Put $n = -x + 3y$. Then $n \in \mathbb{Z}$ and $k = 5n$. Since $5n \in \mathbb{N}$ and $5 \in \mathbb{N}$, we have that $n \in \mathbb{N}$. So $n \geq 1$. Hence $5n \geq 5$. Thus $k \geq 5$.

This shows that 5 is indeed the smallest element in $S$: 5 is in $S$ and $5 \leq k$ for every element $k$ in $S$.

Thus $\text{gcd}(-5, 15) = 5$. \qed