1. Let $E \subseteq \mathbb{R}$ with $m^*(E) < \infty$.

(a) Prove that for all $\epsilon > 0$, there exists an open set $O$ containing $E$ such that $m(O) - m^*(E) < \epsilon$.

(b) Prove that there exists a $G_\delta$-set $G$ containing $E$ such that $m(G) = m^*(E)$.

**Proof:** (a) Let $\epsilon > 0$. Since $m^*(E) < \infty$, it follows from the definition of the outer measure that there exists a countable collection of intervals $\{I_k\}_{k \geq 1}$ of $E$ with

$$\sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon$$

Put $O = \bigcup_{k \geq 1} I_k$. Then $O$ is open and $E \subseteq \bigcup_{k \geq 1} I_k = O$ since $\{I_k\}_{k \geq 1}$ is a ccoi of $E$. By countable subadditivity and the fact that $m(I_k) = l(I_k)$ for all $k \geq 1$, we get that

$$m(O) = m(\bigcup_{k \geq 1} I_k) \leq \sum_{k=1}^{\infty} m(I_k) = \sum_{k=1}^{\infty} l(I_k) < m^*(E) + \epsilon$$

Since $m^*(E) < \infty$, it follows that $m(O) - m^*(E) < \epsilon$.

(b) Let $k \geq 1$. It follows from (a) that there exists an open set $O_k$ containing $E$ with

$$m(O_k) - m^*(E) < \frac{1}{k}$$

Put $G = \cap_{k \geq 1} O_k$. Then $G$ is of type $G_\delta$ and $E \subseteq G$ since $E \subseteq O_k$ for all $k \geq 1$.

Let $n \geq 1$. Then $E \subseteq G = \cap_{k \geq 1} O_k \subseteq O_n$. By monotonicity, we get that

$$m^*(E) \leq m(G) \leq m(O_n) < m^*(E) + \frac{1}{n}$$

So

$$m^*(E) \leq m(G) < m^*(E) + \frac{1}{n} \quad \text{for all } n \geq 1$$

Taking the limit as $n \to +\infty$, we get that

$$m^*(E) \leq m(G) \leq m^*(E)$$

So $m(G) = m^*(E)$.

2. Let $E \subseteq \mathbb{R}$. Prove that $E$ is measurable if and only if for all $\epsilon > 0$, there exist a closed set $F$ and an open set $O$ such that $F \subseteq E \subseteq O$ and $m(O \setminus F) < \epsilon$.

**Proof:** Suppose first that $E$ is measurable. Let $\epsilon > 0$. By Proposition 2.28 (i)(ii)(iv), there exist a closed set $F$ and an open set $O$ such that $F \subseteq E \subseteq O$ and $m(O \setminus E) < \frac{\epsilon}{2}$ and $m(E \setminus F) < \frac{\epsilon}{2}$. Since $O \setminus F = (O \setminus E) \cup (E \setminus F)$, it follows from countable subadditivity that

$$m(O \setminus F) = m((O \setminus E) \cup (E \setminus F)) \leq m(O \setminus E) + m(E \setminus F) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Suppose next that for all $\epsilon > 0$, there exist a closed set $F$ and an open set $O$ such that $F \subseteq E \subseteq O$ and $m(O \setminus F) < \epsilon$. Let $\epsilon > 0$. Let $F$ be a closed set and $O$ an open set such that $F \subseteq E \subseteq O$ and $m(O \setminus F) < \epsilon$.

Since $O \setminus E \subseteq O \setminus F$, it follows from monotonicity that

$$m^*(O \setminus E) \leq m(O \setminus F) < \epsilon$$

So $E$ is measurable by Proposition 2.28(i)(ii). \qed
3. Show that the condition $\mu(E) < +\infty$ is needed in Proposition 2.29 on page 35 (Littlewood’s First Principle).

Solution: Let $E = \bigcup_{n=0}^{\infty} [2n, 2n+1]$. Put $\mu = 1$ (in fact, any positive real number will do the job since we will show that $\mu(\Omega \Delta E) = +\infty$ for any finite union of open intervals $\Omega$). Let $I_1, I_2, \ldots, I_k$ be open intervals. Put $\Omega = \bigcup_{j=1}^{k} I_j$.

Suppose first that $I_j$ is an infinite interval for some $1 \leq j \leq k$, say $I_j = (a, b)$, where $a = -\infty$ or $b = +\infty$.

If $a = -\infty$ then $(-\infty, -M) \subseteq \Omega \setminus E \subseteq E \Delta \Omega$ for some $M > 0$ and so $+\infty = \mu((-\infty, -M)) \leq \mu(E \Delta \Omega)$. If $b = +\infty$ then $\cup_{n=0}^{\infty} [2n-1, 2n] \subseteq \Omega \setminus E \subseteq E \Delta \Omega$ for some $N \in \mathbb{N}$ and so $+\infty = \mu(\cup_{n=0}^{\infty} [2n-1, 2n]) \leq \mu(E \Delta \Omega)$.

Suppose next that $I_j$ is finite for all $1 \leq j \leq k$. Then there exists a positive integer $N$ such that $\Omega \subseteq (-2N, 2N)$. Hence $\cup_{n=N}^{\infty} (2n, 2n+1) \subseteq E \setminus \Omega \subseteq E \Delta \Omega$. So $+\infty = \mu(\cup_{n=N}^{\infty} (2n, 2n+1)) \leq \mu(E \Delta \Omega)$.

In all cases, we have that $\mu(E \Delta \Omega) = +\infty \geq 1 = \epsilon$. \hfill \Box

------------------

4. Let $0 < \alpha < 1$. Let $F_\alpha$ be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the $n$-th deletion stage has length $\frac{\alpha}{3^n}$.

(a) Show that $F_\alpha$ is closed.

(b) Find $\mu(F_\alpha)$.

Solution: For step 1, we remove from $C_0 := [0, 1]$ an open interval of length $\frac{\alpha}{3}$ and end up with a closed set $C_1$ with $\mu(C_1) = 1 - \frac{\alpha}{3}$. For step 2, we remove from $C_1$ two open intervals of length $\frac{\alpha}{3^2}$ and end up with a closed set $C_2$ with $\mu(C_2) = 1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2}$.

For step 3, we remove from $C_2$ four open intervals of length $\frac{\alpha}{3^3}$ and end up with a closed set $C_3$ with $\mu(C_3) = 1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2} - \frac{4\alpha}{3^3}$.

In general, after step $n$, we end up with a closed set $C_n$ with $\mu(C_n) = 1 - \frac{\alpha}{3} - \frac{2\alpha}{3^2} - \frac{4\alpha}{3^3} - \cdots - \frac{2^{n-1}\alpha}{3^n} = 1 - \sum_{k=0}^{n-1} \frac{\alpha}{3} \left(\frac{2}{3}\right)^k$.

Then $F_\alpha = \cap_{n \geq 1} C_n$.

(a) Since $C_n$ is closed for all $n \geq 1$, we see that $F_\alpha$ is the intersection of closed sets. So $F_\alpha$ is closed.

(b) Note that $C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$ and $\mu(C_1) < 1$. By the continuity of the measure (Proposition 2.23(b)), we get that $\mu(F_\alpha) = \mu(\cap_{n \geq 1} C_n) = \lim_{n \to +\infty} \mu(C_n)$

Note that $\lim_{n \to +\infty} \mu(C_n) = \lim_{n \to +\infty} \left(1 - \sum_{k=0}^{n-1} \frac{\alpha}{3} \left(\frac{2}{3}\right)^k \right) = 1 - \sum_{k=0}^{+\infty} \frac{\alpha}{3} \left(\frac{2}{3}\right)^k = 1 - \frac{\alpha}{3} \frac{1}{1 - \frac{2}{3}} = 1 - \alpha$

So $\mu(F_\alpha) = 1 - \alpha$. \hfill \Box
5. We give the definition of a dense set:

If \( A \subseteq B \subseteq \mathbb{R} \) then \( A \) is dense in \( B \) if for all \( x, y \in B \) with \( x < y \) there exists \( a \in A \) such that \( x < a < y \).

Let \( C \) be the Cantor set. Prove that \([0, 1] \setminus C\) is dense in \([0, 1]\).

**Proof**: Let \( x, y \in [0, 1] \) with \( x < y \).
Suppose that \((x, y) \cap ([0, 1] \setminus C) = \emptyset\). Since \((x, y), C \subseteq [0, 1]\), we get that \((x, y) \subseteq C\). By monotonicity, we find that \(0 < y - x = l((x, y)) = m((x, y)) \leq m(C)\), a contradiction since \(m(C) = 0\).
Hence \((x, y) \cap ([0, 1] \setminus C) \neq \emptyset\). Let \( a \in (x, y) \cap ([0, 1] \setminus C) \). Then \( a \in [0, 1] \setminus C \) and \( x < a < y \) since \( a \in (x, y)\).
So \([0, 1] \setminus C\) is dense in \([0, 1]\).

\[ \square \]

6. Let \( A \subseteq \mathbb{R} \) be bounded with \( m^*(A) > 0 \). Let \( C_A \) be a choice set for the rational equivalence relation on \( A \). Let \( E \) be a measurable subset of \( C_A \). Prove that \( m(E) = 0 \).

**Proof**: We use the same notations as in the proof of Theorem 2.25. We showed that
\[
A \subseteq \bigcup_{n=1}^{\infty} (C + q_n) \subseteq [-3N, 3N]
\]
Since \( E \subseteq C_A \subseteq A \), we get that
\[
\bigcup_{n=1}^{\infty} (E + q_n) \subseteq \bigcup_{n=1}^{\infty} (C + q_n) \subseteq [-3N, 3N]
\]
It follows from monotonicity that
\[
m^* \left( \bigcup_{n=1}^{\infty} (E + q_n) \right) \leq m([-3N, 3N]) = 6N < +\infty \quad (*)
\]
Since \( E \) is measurable, we have that \( E + q_n \) is measurable for all \( n \in \mathbb{N} \) by Proposition 2.22. Using countable additivity and the fact that the outer measure is translation-invariant, we get
\[
m \left( \bigcup_{n=1}^{\infty} (E + q_n) \right) = \sum_{n=1}^{\infty} m(E + q_n) = \sum_{n=1}^{\infty} m(E) = \begin{cases} 0 & \text{if } m(E) = 0 \\ +\infty & \text{if } m(E) > 0 \end{cases} \quad (**) \]
Combining \((*)\) and \((**)*\), we see that \( m(E) = 0 \).

**Remark**: There is an easier proof by noting the following:

If \( C \) is a choice set for some set \( A \), then any subset of \( C \) is a choice set for itself.

Indeed, let \( C \) be a choice set for some set \( A \). Then \( C \) contains exactly one element from each equivalence class on \( A \) of the rational equivalence relation. So any two distinct elements of \( C \) are not related. So if \( D \subseteq C \) then the equivalence classes on \( D \) are all singletons and \( D \) is its own choice set (in fact, the only choice set of \( D \)).

So let \( E \subseteq C_A \) be measurable. Suppose that \( m(E) > 0 \). Since \( E \) is bounded and is a choice set for \( E \), it follows from Theorem 2.25 that \( E \) is non-measurable, a contradiction. Hence \( m(E) = 0 \).